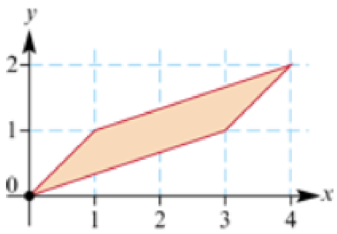
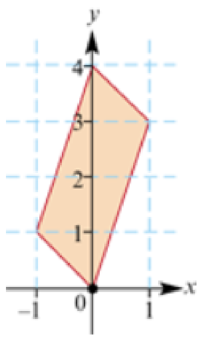
Exercise 16G: Solutions

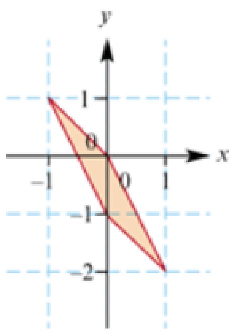
1 a The area will be given by
$|\det B| = |3 \times 1 - 1 \times 1| = |2| = 2.$



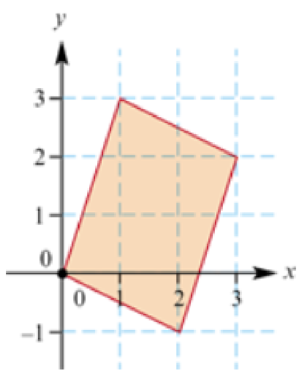
b The area will be given by
$|\det B| = |(-1) \times 3 - 1 \times 1| = |-4| = 4.$



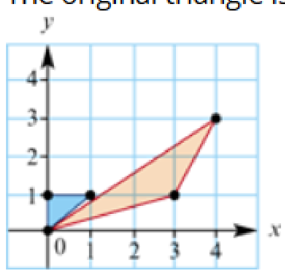
c The area will be given by
$|\det B| = |1 \times 1 - (-1) \times (-2)| = |-1| = 1.$



d The area will be given by
$|\det B| = |2 \times 3 - 1 \times (-1)| = |6 + 1| = 7.$



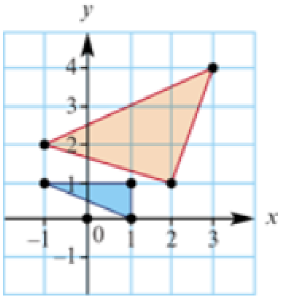
2 a The original triangle is shown in blue, and its image is in red.



b The area of the original triangle is $\frac{1}{2}$. Therefore the area of the image will be given by,

$$\begin{aligned}\text{Area of Image} &= |\det B| \times \text{Area of Region}\\ &= |1 \times 1 - 3 \times 2| \times \frac{1}{2}\\ &= |-5| \times \frac{1}{2}\\ &= 2.5.\end{aligned}$$

3 a The original triangle is shown in blue, and its image is in red.



b The area of the original triangle is 1. Therefore the area of the image will be given by,

$$\begin{aligned}\text{Area of Image} &= |\det B| \times \text{Area of Region}\\ &= |2 \times 3 - 1 \times 1| \times 1\\ &= 5.\end{aligned}$$

4 Since the original area is 1 and the area of the image is 6, we have,

$$\begin{aligned}|\det B| \times 1 &= 6\\ |m \times m - 2 \times (-1)| &= 6\\ |m^2 + 2| &= 6\\ m^2 + 2 &= 6 \text{ (since } m^2 + 2 > 0)\\ m^2 &= 4\\ m &= \pm 2.\end{aligned}$$

5 The original area is 1 and the area of the image is 2. Therefore,

$$\begin{aligned}\text{Area of Image} &= |\det B| \times \text{Area of region}\\ 2 &= |m \times m - m \times 1| \times 1\\ 2 &= |m^2 - m|\end{aligned}$$

Therefore, either

$$m^2 - m = 2 \text{ or } m^2 - m = -2.$$

In the first case, we have

$$\begin{aligned}m^2 - m - 2 &= 0\\ (m-2)(m+1) &= 0\\ m &= -1, 2.\end{aligned}$$

In the second case, we have

$$m^2 - m + 2 = 0.$$

This has no solutions since the discriminant of the quadratic equation is $\begin{aligned}\Delta &= b^2 - 4ac\\ &= 1^2 - 4(1)(2)\\ &= 1 - 8 < 0.\end{aligned}$

6 a i If

$$B = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

then

$$|\det B| = |1 \times 1 - k \times 0| = 1.$$

ii If

$$B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

then

$$\begin{aligned} |\det B| &= |\cos\theta\cos\theta - (-\sin\theta)\sin\theta| \\ &= |\cos^2\theta + \sin^2\theta| \\ &= 1. \end{aligned}$$

iii If

$$B = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

then

$$\begin{aligned} |\det B| &= |\cos 2\theta(-\cos 2\theta) - \sin 2\theta \sin 2\theta| \\ &= |-(\cos^2 2\theta + \sin^2 2\theta)| \\ &= |-1| \\ &= 1 \end{aligned}$$

b i This transformation is a dilation by a factor k away from the y-axis and afactor of $\frac{1}{k}$ away from the x-axis.

ii We have, $\begin{aligned} |\det B| &= |k \times \frac{1}{k} - 0 \times 0| \\ &= 1 \end{aligned}$

7 a We have,

$$\begin{aligned} |\det B| &= |x \times (x+2) - 1 \times (-2)| \\ &= |x^2 + 2x + 2| \\ &= |(x^2 + 2x + 1) + 1| \end{aligned}$$

$$(\text{completing the square})$$

$$\begin{aligned} &= |(x+1)^1 + 1| \\ &= (x+1)^2 + 1. \end{aligned}$$

b The area will be a minimum at the turning point of the parabola whose equation is $y = (x+1)^2 + 1$. This occurs when $x = -1$.

8

We require that $\begin{aligned} |\det B| &> 2 \\ |4m - 6| &> 2. \end{aligned}$

Therefore, either $4m - 6 > 2$ or $4m - 6 < -2$. In the first case, $m > 2$. In the second case, $m < 1$. Therefore $m > 2$ or $m < 1$.

9 Since $(1,0) \to (1,0)$ we can assume that the matrix is of the form $\begin{bmatrix} 1 & b \\ 0 & c \end{bmatrix}$. Since the area is $\frac{1}{2}$, we know that

$$\begin{aligned} |1 \times c - b \times 0| &= \frac{1}{2} \\ |c| &= \frac{1}{2} \\ c &= \pm\frac{1}{2} \end{aligned}$$

Since $(0,1) \to (b,c)$, one corner of the rhombus will be given by the second column (written as a coordinate). Moreover, since it is a rhombus, the distance from $(0,0)$ to (b,c) is 1. Therefore,

$$\begin{aligned} b^2 + c^2 &= 1^2 \\ b^2 + \left(\frac{1}{2}\right)^2 &= 1 \end{aligned}$$

$$b^2 = \frac{3}{4}$$
$$b = \pm\frac{\sqrt{3}}{2}$$

so that the required matrix is

$$\begin{bmatrix} 1 & \pm\frac{\sqrt{3}}{2} \\ 0 & \pm\frac{1}{2} \end{bmatrix}.$$

10a We can assume that $(1,0) \to (a,c)$ and $(0,1) \to (b,d)$. Therefore, the required matrix is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

b The area of the original triangle is $\frac{1}{2}$. Therefore, the area of the image will be given by,

$$\begin{aligned} \text{Area of Image} &= |\det B| \times \text{Area of Region} \\ &= |a \times d - b \times c| \times \frac{1}{2} \\ &= \frac{1}{2}|ad - bc| \end{aligned}$$

c If a, b, c, d are all rational numbers then so too is $\frac{1}{2}|ad - bc|$.

d We will assume that the triangle has vertices $O(0,0)$, $A(a,c)$ and $B(b,d)$.Then the area of the triangle is

$$\frac{1}{2}|ad - bc|. \quad (1)$$

We will find another expression for the area. Since the triangle is equilateral,

$$OB = OA = \sqrt{a^2 + c^2}$$

Using Pythagoras' Theorem, we can show that

$$\begin{aligned} MB^2 + OM^2 &= OB^2 \\ MB^2 + \left(\frac{1}{2}OA\right)^2 &= OA^2 \\ MB^2 + \frac{1}{4}OA^2 &= OA^2 \\ MB^2 &= \frac{3}{4}OA^2 \\ MB &= \frac{\sqrt{3}}{2}OA \\ &= \frac{\sqrt{3}\sqrt{a^2 + c^2}}{2}. \end{aligned}$$

Therefore, another expression for the area is

$$\begin{aligned} A &= \frac{1}{2} \times OA \times MB \\ &= \frac{1}{2} \times \sqrt{a^2 + c^2} \times \frac{\sqrt{3}\sqrt{a^2 + c^2}}{2} \\ &= \frac{\sqrt{3}(a^2 + b^2)}{4} \quad (2) \end{aligned}$$

Equating equations (1) and (2) gives,

$$\frac{\sqrt{3}(a^2+b^2)}{4} = \frac{1}{2}|ad-bc|$$

$$\sqrt{3} = \frac{2|ad-bc|}{a^2+b^2}$$

Since a, b, c and d are all rational numbers, the right hand side of the above expression is rational. This contradicts the fact that $\sqrt{3}$ is irrational.